

Wee partons in large nuclei: from virtual dream to hard reality *

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1. Introduction

The title of this talk refers to wee partons in large nuclei. The first part of this section will be a brief discussion of the theory and phenomenology of wee parton distributions in QCD. The latter part of this section (and subsequent sections) will be about wee partons distributions in large nuclei and their relevance to experiments at RHIC and LHC.

1.1. A brief introduction to the low x problem in QCD

One of the more interesting problems in QCD today is to understand the space-time (or equivalently, momentum) distributions of “wee partons” in hadrons and nuclei. The phrase “wee parton” was coined by Richard Feynman [1] to describe those partons which carry a very small fraction of the longitudinal momentum of the parent hadron or nucleus. This momentum fraction is the Feynman x variable and for wee partons, $x \ll 1$. To lowest order in the coupling constant, the Feynman x variable is equal to the well

known Bjorken x of deeply inelastic scattering experiments.

Why is this problem of wee parton distributions interesting? To understand this let us first briefly review the usual Dokshitzer–Gribov–Lipatov–Altarelli–Parisi branching process in perturbative QCD [2]. This branching process describes a virtual, ladder-like cascade, where partons with higher values of transverse momentum k_t and x , “split” into softer partons with lower values of k_t and x and so on along the ladder. Each rung in the ladder provides logarithmic contributions to the cross section of the sort

$$\alpha_S \int \frac{d^2 k_t}{k_t^2} \int \frac{dx}{x}. \quad (1)$$

In the Double Leading Logarithmic Approximation (DLA), both x and k_t are ordered along the ladder; this is equivalent to summing leading logs in both variables. In the DLLA, the gluon distribution function, for $x \ll 1$, grows as [3]

$$x \frac{dN}{dx}(x, Q^2; Q_0^2) \sim \exp \left(2\sqrt{\log(1/x) \log \log(Q^2/Q_0^2)} \right), \quad (2)$$

where Q_0^2 and Q^2 are the respectively the lowest and highest values of k_t^2 along the

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rungs of the ladder. The latter value is usually identified with the momentum transfer squared in deeply inelastic scattering experiments while the former typically sets the factorization scale.

However, if x is very small and Q^2 is not large enough, the leading logarithmic terms in x will dominate and it may be appropriate to sum those alone. In this approximation, known as the BFKL approximation [4] (after the initials of the authors), one has ordering along the ladder rungs only in x . In the Regge–Mueller language, this corresponds to a t -channel exchange of a “perturbative pomeron” in the virtual photon+ target scattering process. The gluon distribution then has the form

$$x \frac{dN}{dx}(x, Q^2; Q_0^2) \sim \sqrt{\frac{Q^2}{Q_0^2}} \frac{1}{x^\lambda}, \quad (3)$$

where $\lambda = 4\alpha_s \log(2)$. This behaviour of the gluon distributions is often called the Lipatov enhancement. An interesting feature about the BFKL approach is that, unlike the DLLA, it is *not* a twist expansion (as in the Operator Product Expansion).

Regardless of whether the DLLA or the BFKL approximations are more appropriate for the x values of interest, they have one outstanding feature in common: for $x \ll 1$, the density of wee partons (dN/dx) grows rapidly. In both cases, it grows faster than $1/x$. At sufficiently small values of x , one enters a regime where partons from neighbouring ladders overlap spatially. This regime of “high density” QCD [5] is of interest because it

describes the many body behaviour of the quanta of the fundamental theory.

One consequence of the overcrowding of partons is that two soft partons may recombine to form a harder parton. Another consequence of the overcrowding is the screening of parton–parton interactions by the cloud of surrounding wee partons [6]. Both of these processes may inhibit the growth of parton distributions and cause them to saturate at some critical value of x . Incidentally, the saturation of parton distributions as $x \rightarrow 0$ is required by unitarity—the so-called Froissart bound. Because these many body phenomena are very complex, it is especially gratifying to realize that they may (to a degree) be understood using well known, weak coupling techniques [7].

Why are weak coupling techniques applicable in this problem? The only scale in the problem at low x is ρ , the density of partons. The coupling constant α_S will run as a function of this scale and if $\Lambda_{QCD} \ll \rho$, then $\alpha_S(\rho) \ll 1$. *This qualitative argument can be proven rigorously and is the basis of all that follows.*

Thus far, we have focused on some of the theoretical reasons why one must better understand wee parton distributions. These theoretical impulses have been around from the pre-history of QCD. What is re-generating interest in a previously moribund field is a new generation of experiments, which have begun to probe the high density phase of QCD. Foremost among these is HERA, the electron–proton collider experiment at DESY [8]. At HERA, x ’s as low as $x \sim 10^{-5}$ and

Q^2 as high as $Q^2 \sim 10^5 \text{ GeV}^2$ are attained. One of the observables measured at HERA is $F_2^{ep}(x, Q^2)$, the electromagnetic form factor of the proton. At the above mentioned values of x , F_2 continues to rise with decreasing x , for a wide range of Q^2 values.

Since F_2 can be related directly to the sea quark distributions and indirectly to the gluon distributions, these experimental results seem to confirm the behaviour predicted by the theoretical models discussed above. The data are apparently not good enough to distinguish between the DLA and BFKL scenarios—this issue is controversial and will hopefully be settled conclusively in the near future.

Another class of experiments which probe the region of low x are the deeply inelastic scattering experiments at Fermilab (E665) and CERN (NMC). These experiments are fixed target experiments where a high energy lepton scatters off a nuclear target. It is observed [9] that the ratio F_2^A/AF_2^N shows a significant depletion at $x \leq 10^{-2}$ —a phenomenon known as “shadowing”. The shadowing of nuclear distributions at low x shows an unambiguous A dependence of $A^{0.9}$. Can shadowing be understood as a consequence of the faster (i.e., at higher x) saturation of nuclear distributions as opposed to hadron distributions? Naively, the answer is yes, but a conclusive answer will have to be a quantitative one.

Finally, there are the collider experiments at RHIC and LHC which will also probe fairly low x 's ($\sim p_t/\sqrt{s}$). The wee parton distributions of the nuclei will pro-

vide the initial conditions for the evolution of the system into a quark gluon plasma. Further, most of the observables in a nuclear collision will depend sensitively on the wee parton distributions. In the following sub-section we will discuss briefly the problem of wee parton distributions in large nuclei. Subsequent sections will address the problem in greater detail.

1.2. Wee partons in large nuclei: an outline

In recent papers [10,11], Larry McLerran and I have argued that large nuclei are an excellent theoretical tool to study the low x problem. We show that for large nuclei, in the range of $x \ll A^{-1/3}$, the large density of partons introduces a scale $\mu^2 \approx A^{1/3} \text{ fm}^{-2}$, which can be identified as the average *valence quark* color charge squared per unit area. When μ^2 is large (A is large) the coupling constant $\alpha_s(\mu^2)$ is small and a weak coupling expansion can be employed to compute quark and gluon distributions.

On the phenomenological side, we expect that this first principles calculation of quark and gluon distributions in QCD will help clarify our understanding of the initial stages of nuclear collisions at RHIC and LHC. There exist state of the art partonic cascade models which provide an excellent initial orientation to this problem [12]. However, since cascade models do not accurately simulate the wee parton sea, the enhancement of the momentum scale due to the large parton density will, in comparison, likely lead to i) higher temperatures, ii) enhanced minijet

production and iii) an enhanced contribution of “intrinsic” strangeness and charm, than those predicted in these models.

We will now discuss how the paper is organized. We formulate the problem of low x parton distributions as a many body problem in the presence of an external source. The source here is a sheet of static, valence quarks localized in the longitudinal direction but infinite in extent and uniformly varying in the transverse direction. Next, we write down a partition function for this system which includes a stochastic averaging over the external sources of color charge. This procedure introduces an additional dimensionful parameter μ^2 in the theory. It is defined as the average valence quark colour charge squared per unit area.

The approach we adopt to solve the above mentioned problem is as follows: we first obtain the classical background field in the presence of the source. It turns out that there is a very simple solution to the classical field equations in which the only non-zero fields are the transverse vector fields which are zero ahead of the source and two dimensional pure gauge fields behind the source. The gluon distribution functions in the background field can thus be expressed as correlation functions of a two dimensional, Euclidean field theory.

For parton transverse momenta $\alpha_s^2 \mu^2 \ll k_t^2$, which we identify as a weak coupling regime, the gluon distributions obey the well known Weizsäcker–Williams distribution scaled by $\mu^2 \sim A^{1/3}$. In the strong coupling regime $k_t^2 \ll \alpha_s^2 \mu^2$, the distribution func-

tions still vary as $1/x$ but the transverse momentum dependence is changed. This likely reflects the sharp decay of correlation functions in transverse space. We have not succeeded in obtaining an analytical expression for the transverse momentum dependence—but have succeeded instead in writing an algorithm to solve the equations numerically using a Monte Carlo procedure.

We next outline the procedure to compute small fluctuation Green’s functions for scalars, vectors and fermions in this classical background field. The detailed computations, performed in collaboration with A. Ayala, J. Jalilian–Marian and L. McLerran are reported elsewhere [13–15]. These Green’s functions are useful when one computes the higher order contributions in α_s to the quark and gluon distribution functions. The fermion Green’s functions are used to determine the sea quark distributions. We find that the enhanced momentum scale arising from the higher density of partons leads to an enhancement of the strange and charm quark contribution to the nuclear wavefunction.

We then proceed to discuss briefly higher order corrections to our results – specifically, the one loop corrections to the classical background field and the origin of the Lipatov enhancement in our approach. We also speculate how our solutions may be iterated to all orders in α_s . Finally, we discuss recent progress in extending our approach to nuclear collisions and end our presentation with a brief summary.

2. Formulation of the problem

In this section we will formulate the problem of calculating the distributions of partons in the nuclear wavefunction as a well defined many body problem. To do this we will work in the Infinite Momentum frame and use the technique of Light Cone quantization. We will also be working in Light Cone gauge $A^+ = 0$. There are several advantages to these choices. One of these is the fact that it is only in Light Cone gauge that partons have a manifest physical interpretation as the quanta of the theory. Further, if we work on the Light Cone, it is possible to construct a simple and intuitive Fock space basis [16] on which our formalism relies heavily. One consequence is that the electromagnetic form factor of the hadron F_2 , which is measured in deeply inelastic scattering experiments, is simply related to parton distributions by the relation [17]

$$F_2(x, Q^2) = \langle \int^{Q^2} d^2k_t dk^+ x \delta(x - \frac{k^+}{P^+}) \sum_{\lambda=\pm} a_\lambda^\dagger a_\lambda \rangle \quad (4)$$

In the above, P^+ is the momentum of the nucleus, $k^+ = (k + k_z)/\sqrt{2}$ and k_t are the parton “longitudinal momentum” and transverse momenta respectively, Q^2 is the momentum transfer squared from the projectile and $a^\dagger a$ is the number density of partons in momentum space. Hence, in principle one only needs up to integrate the calculated distributions to the scale Q^2 to make comparison with experiment. One difficulty in the Light Cone formalism is the problem of choosing appropriate boundary conditions. This issue is beyond

the scope of this talk. For an alternative approach to the problem, the reader may consult the recent paper by Makhlin [18].

2.1. A static source on the Light Cone

We begin by making several physically plausible assumptions. First, if the density of partons per unit area at low x is large, and weak coupling techniques are applicable, the recoil experienced by the valence quarks due to bremsstrahlung of low x partons is small—the valence quarks obey straight line trajectories. We can thus replace the valence quarks in the infinite momentum frame nuclear wavefunction by static, external sources of charge. Next, if we require that the nucleus be Lorentz contracted to a size which is much smaller than the wavelength of the parton in a frame co-moving with its longitudinal momentum, then

$$2Rm/P \ll 1/xP \implies x \ll 1/Rm, \quad (5)$$

where R is the nuclear radius, P the momentum of the nucleus and m the nucleon mass. From the above inequality, we deduce that when $x \ll A^{-1/3}$, the partons see a sheet of static color charges of large transverse extent but localized in the longitudinal direction. Since the average transverse momentum scale $k_\perp \gg 1/R$, the valence quark distribution can be taken to be uniform and infinite in extent in the transverse direction.

It is appropriate to use Light Cone coordinates to investigate the dynamics of the infinite momentum frame wavefunction of the nucleus. The conventions used

here are discussed in Ref. [10]. In the Light Cone formalism, the “external” current experienced by the low x partons due to the static, valence quark charges takes the form

$$J_a^\mu = \delta^{\mu+} \rho_a(x^+, \vec{x}_\perp) \delta(x^-). \quad (6)$$

In the above, $x^\pm = (t \pm x)/\sqrt{2}$, where x^+ is the Light Cone time.

The Light Cone Hamiltonian P^- (which generates translations in x^+), in the presence of an external source, can be written as

$$P^- = \frac{1}{4} F_t^2 + \frac{1}{2} (\rho_F + D_t \cdot E_t) \frac{1}{P_{+2}} (\rho_F + D_t \cdot E_t) + \frac{1}{2} \psi^\dagger (M - \not{P}_t) \frac{1}{P_{+2}} (M + \not{P}_t) \psi.$$

Here ρ_F is the charge density due to the external source plus the dynamical quarks, A_t and ψ are the dynamical vector and spinor fields, $E_t = \partial_- A_t$ is the transverse electric field and F_t^2 is the transverse field strength tensor. In writing the above expression, we have made use of the constraint equations on the Light Cone to eliminate the non-dynamical fields.

2.2. Ground state expectation values in the presence of external sources

In QED, the infinite momentum frame wavefunction of the system with the external source in Eq. (6) is a coherent state [10]. We have not succeeded in doing the same in QCD and have concentrated instead on computing ground state expectation values in the presence of external sources. The partition function for

the ground state of the low x partons in the presence of the valence quark external source is

$$Z = \langle 0 | e^{iTP^-} | 0 \rangle = \lim_{T \rightarrow i\infty} \sum_N \langle N | e^{iTP^-} | N \rangle_Q. \quad (8)$$

The sum above also includes a sum over the color labels of the source of color charge (denoted by Q) generated by the valence quarks.

In principle, for quantized sources of color charge, evaluating the trace over different values of the color charge is difficult. However, for a large nucleus, the problem can be simplified considerably. Visualize the transverse space as a grid of boxes of size $d^2 x_t \gg 1/\rho_{valence} \sim A^{1/3}$ or equivalently, of parton transverse momenta $q_t^2 \ll A^{1/3} \text{ MeV}^2$. In this kinematic range, the number of valence quarks, and therefore the charge Q in each box, will be much greater than one. A large number of charges corresponds to a higher dimensional representation of the color algebra—the sum of the color charges of the valence quarks in a grid can be treated classically.

Further, if the total charge in the grid is much less than the maximum possible charge the central limit theorem tells us that the density of states corresponding to charge Q is a Gaussian $e^{-Q^2/2\mu^2}$. Summing over the color labels of the states is therefore equivalent to introducing in the path integral the measure

$$\int [d\rho] \exp \left(- \frac{1}{2\mu^2} \int d^2 x_t \rho^2(x_t) \right). \quad (9)$$

This ensures that

$$\langle \rho^a(x_t) \rho^b(y_t) \rangle = \mu^2 \delta^{ab} \delta^{(2)}(\vec{x}_t - \vec{y}_t); \quad \langle \rho^a(x_t) \rangle = 0. \quad (10)$$

Above, the variance μ^2 is the average valence quark color charge squared per unit area. It can be written as

$$\mu^2 = \rho_{valence} < Q^2 > \equiv \frac{3A}{\pi R^2} \frac{4}{3} g^2 \sim 1.1 A^{1/3} fm^{-2} \quad (11)$$

where $< Q^2 > = 4g^2/3$ is the average charge squared of a quark.

We can now write the path integral representation of the partition function Z in the Light Cone gauge $A_- = 0$ as

$$Z = \int [dA_t dA_+][d\psi^\dagger d\psi][d\rho] \exp \left(iS + ig \int d^4x A_+(x) \delta(x^-) \rho(x) - \frac{1}{2\mu} \int d^2x_\perp \rho^2(x_\perp, x_t) \right) \quad (13)$$

Hence, the result of our manipulations is to introduce a dimensionful parameter $\mu^2 \approx 1.1 A^{1/3} fm^{-2}$ in the theory. If we impose the current conservation $D_\mu J^\mu = 0$ and integrate over the external sources ρ , we obtain an expression [10] containing modified propagators and vertices.

We shall however attempt to solve the above many body problem in the following fashion: we first solve for the classical background field, compute the small fluctuation determinant in the background field to obtain Green's functions and propagators, use these to compute the first radiative corrections to the background field and systematically iterate the process to all orders.

3. The Classical Background Field

We first discuss the problem of computing the solution of the classical equations of motion for the gluon field in the presence of a source which is a delta function

along the Light Cone. The second part of this discussion is about computing correlation functions in this background field. Notice from Eq. (12) that there is still an integral over ρ with a Gaussian weight to average over at the end. At each order in α_S , this procedure is equivalent to summing the theory to all orders in the effective coupling $\alpha_S \mu/k_t$.

3.1. Solutions to the Classical Equations

The equations of motion are

where J , the classical Light Cone source, may be represented as

$$J_a^\mu = \delta^{+\mu} Q_a(x^+, x_t) \delta(x^-). \quad (14)$$

We will work in Light Cone gauge where $A_- = -A^+ = 0$.

There exists a solution of the equations of motion for this problem, where the longitudinal component A_+ , which is not zero by a gauge, vanishes by the equations of motion

$$A_+ = -A^- = 0. \quad (15)$$

The only non-zero components of the field strength are the transverse components which we require to be of the form

$$A_i(x) = \theta(x^-) \alpha_i(x_t). \quad (16)$$

This functional form is what we expect for a classical field generated by a source traveling close to the speed of light with $x = t$. For $x > t$, the source has not yet arrived, and for $x < t$, the source should produce a field.

If we further require that $F^{ij} = 0$ (where i and j are transverse components), we see that we have a solution of the equations of motion as long as

$$\nabla \cdot \alpha = g\rho(x_t). \quad (17)$$

Here ρ is the surface charge density associated with the current J . There is no dependence on x^- because we have factored out the delta function. The dependence on x^+ goes away because of the extended current conservation law $D_\mu J^\mu = 0$ in the background field.

The condition that $F^{ij} = 0$ is precisely the condition that the field α is a gauge transform of the vacuum field configuration for a two dimensional gauge theory. The field configuration which is a gauge transform of the vacuum field configuration for a two dimensional field theory may be written as

$$\tau \cdot \alpha_i = -\frac{1}{ig} U \nabla_i U^\dagger. \quad (18)$$

The dependence of the U fields on the surface charge density ρ is then given by the relation

$$\nabla \cdot U \nabla U^\dagger = -ig^2 \rho(x_t). \quad (19)$$

We have not been able to construct analytical solutions for the above equation for arbitrary dependence of the surface charge density on x_t . However, recently we have found a numerical algorithm which will solve the above equation on a 2-D lattice using a Monte Carlo procedure [19].

3.2. Computing Correlation Functions

To compute correlation functions associated with our classical solutions, we

must integrate over all color orientations of the external sheet of charge. This is equivalent to computing the expectation value of $\langle \alpha_i(x_t) \alpha_j(y_t) \rangle$ with the measure (expressed in terms of the compact fields U)

$$\int [dU] \exp \left(-\frac{1}{g^4 \mu^2} \int d^2 x_t \text{Tr} \left[\vec{\nabla} \cdot \left(U \frac{1}{i} \vec{\nabla} U^\dagger \right) \right]^2 \right) \det(\vec{\nabla} \cdot U)$$

We see that the measure for this theory is that for a two dimensional Euclidean field theory where the expansion parameter is $\alpha_S^2 \mu^2 / k_t^2$. Since the theory is two dimensional, it should be straightforward to compute correlation functions to all orders in $\alpha_S^2 \mu^2 / k_t^2$ using lattice Monte Carlo methods. Further, since the theory is ultraviolet finite, there should be no problems extrapolating to the continuum limit.

Since we can solve Eq. (19) numerically, the correlation functions can still be expressed as an integral over the Gaussian measure in ρ . We then do not need to worry about the nasty Fadeev–Popov term in the above measure.

The correlation function for the computation of the transverse momentum dependence of the structure function may be simply evaluated to be

$$D(k_t) = \frac{1}{\alpha_S} H(k_t) \quad (21)$$

where D is the propagator for the two dimensional theory

$$(2\pi)^2 \delta^2(k_t - q_t) \delta_{ij} D(k_t) = \langle \alpha_i \alpha_j \rangle \quad (22)$$

The relation between distribution functions and propagators is straightforward and is discussed explicitly in Ref. (13). For instance, for a scalar field $\phi^\alpha(x^+, x_t)$, in

the fundamental representation, the distribution function is given by the relation

$$\frac{dN}{d^3k} = 2i \frac{(2k^+)}{(2\pi)^3} \sum_{\alpha} D^{\alpha\alpha}(x^+, \vec{k}, x^+, \vec{k}). \quad (23)$$

The propagator in the above equation is defined as

$$D(x^+, \vec{k}, y^+, \vec{q}) = \int d^3x d^3y e^{-ikx+iqy} G(x, y), \quad (24)$$

where

$$G^{\alpha\beta}(x, y) = -i \langle \phi^{\alpha}(x) \bar{\phi}^{\beta}(y) \rangle. \quad (25)$$

Now, in the correlation functions of the classical field $\langle A_t(x_t) A_t(y_t) \rangle$, the dependence on x^- is only through a step function, and upon Fourier transforming gives only a factor of $1/k^+$. The distribution functions associated with this field, to all orders in $\alpha_S^2 \mu^2$, have the general form [11]

$$\frac{1}{\pi R^2} \frac{dN}{dx d^2k_t} = \frac{(N_c^2 - 1)}{\pi^2} \frac{1}{x} \frac{1}{\alpha_S} H(k_t^2 / \alpha_S^2 \mu^2). \quad (26)$$

When the relevant momentum scale is $k_t^2 \gg \alpha_S^2 \mu^2$, the theory is in the weak coupling region and may be evaluated perturbatively. When $y \rightarrow \infty$,

$$H(y) = 1/y, \quad (27)$$

and we obtain for the distribution function the Weizsäcker–Williams result

$$\frac{1}{\pi R^2} \frac{dN}{dx d^2q_t} = \frac{\alpha_S \mu^2 (N_c^2 - 1)}{\pi^2} \frac{1}{x q_t^2}, \quad (28)$$

scaled by μ^2 .

For small values of $k_t^2 \ll \alpha_S^2 \mu^2$, we are in the strong coupling phase of the theory. In this phase of the theory, we expect that there should be no long range order.

Correlation functions of x_t should die exponentially at large distances, or alternatively the Fourier transform of correlation functions should go like

$$\frac{dN}{d^2k_t} \sim \frac{1}{k_t^2 + \alpha_S \mu^2}, \quad (29)$$

which plateaus off to a constant for small momentum. This is shown in Fig. 1. Hence, $\alpha_S \mu$ is like a Debye scale which guarantees the finiteness of the gluon distribution functions for small momentum.

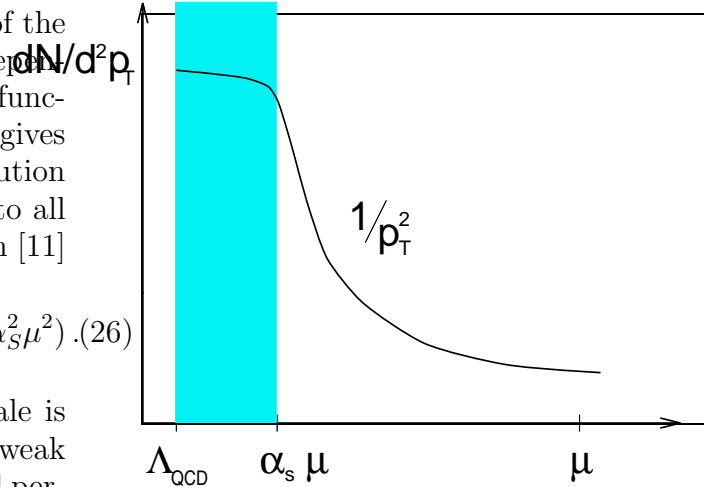


Figure 1. Transverse momentum distribution for Weizsäcker–Williams fields

Note that the gluon distribution function in the weak coupling Weizsäcker–Williams regime is additive in A . If the behaviour in the strong coupling regime would resemble that in Eq. (29), the gluon distribution function would then be proportional to $A^{2/3}$. If we integrate over all k_t^2 to some scale Q^2 , the integrated distribution would probably have a dependence which lies between $A^{2/3}$ and A . One must

keep in mind however, the fact that higher orders will likely modify the A dependence of the classical distribution functions.

4. Small fluctuations and higher orders in perturbation theory

In previous sections, we discussed only the *classical* solutions of the Yang–Mills equations in the presence of the valence quark source defined in Eq. (14). In this section, we will outline a procedure to systematically compute quantum corrections to our background field to all orders. This procedure is exactly equivalent to the familiar Dyson–Schwinger expansion in many body physics [20].

We begin by considering small fluctuations around our classical background,

$$A_{cl} = A_{cl}^0 + \delta A. \quad (30)$$

Substituting this in the partition function in Eq. (12), we keep only terms upto $O(\delta A^2)$ in the action. The small fluctuations propagator may be computed directly from the action or equivalently by substituting Eq. (30) in the Yang–Mills equations, keeping terms linear in δA , and solving the resulting eigenvalue equation.

The gluon Green’s function is then obtained from the general expression

$$G(x, y) = \sum_{\lambda} \frac{\delta A_{\lambda}(x) \delta A_{\lambda}^*(y)}{\lambda}. \quad (31)$$

In Ref. (13), we obtained Green’s functions for scalars, vectors and fermions in the non–Abelian Weizsäcker–Williams (NAWW) background field. It turns out that the gluon Green’s function we computed there in $A^+ = 0$ gauge, is incorrect and is actually the Green’s function

in $A^- = 0$ gauge instead. The direct computation of the Green’s function in Light Cone gauge $A^+ = 0$, is hampered by the presence of the singular source term in the small fluctuation equations of motion. The correct procedure for computing the Light Cone gauge Green’s functions is discussed in Ref. (14). The reader is referred to the papers Ref. (13) and Ref. (14) for relevant details.

The general expression for the two point function to all orders in the coupling constant is given by

$$\langle A_t A_t \rangle = \langle A_t^{cl} \rangle \langle A_t^{cl} \rangle + \langle A_t^{qu} A_t^{qu} \rangle, \quad (32)$$

where $\langle A_t^{cl} \rangle$ is the classical field to all orders and $\langle A_t^{qu} A_t^{qu} \rangle$ is the small fluctuation Green’s function in the background field. The above expectation values also include a stochastic averaging over the sources of external charge with the Gaussian weight discussed in Eq. (9). Recall that as a consequence, there are two expansion parameters in the theory: α_S and $\alpha_S \mu$. So one should be wary of naive α_S counting!

The zeroth order contribution to $\langle A_t^{cl} \rangle$ is the classical background field we discussed in previous sections and is of order $O(1/g) \sum_1^n (\alpha_S \mu)^n$. The lowest order contribution to the distribution functions then gives

$$A_t^{(0)} A_t^{(0)} \propto \frac{1}{g^2} (\alpha_S \mu)^{2n} \longrightarrow \alpha_S \mu^2 (n = 1), \quad (33)$$

which is the result obtained in Eq. (28).

The next order contribution $O(g^0)$ to the distribution function has two components. The first is from the small fluctuation Green’s function $\langle A_t^{qu} A_t^{qu} \rangle$ in Eq. (32) above. Recall that the Green’s functions

we compute are directly related to distribution functions by Eq. (23). In momentum space this is equivalent to both legs of the distribution function having the same momentum. Performing the computation, we find a term proportional to $\log(x)/x$. If we iterate this procedure to all orders (the distribution functions now forming a ladder)—summing the leading logs in x , we might get a series which looks like

$$\frac{1}{x} \left(1 + C\alpha_S \log x + \frac{C^2\alpha_S^2}{2} \log^2 x + \dots \right) \equiv \frac{1}{x^{1+C\alpha_S}} \quad (34)$$

The crucial point is: what is the coefficient C ? How does it depend on μ^2 ? This still remains to be settled: we are working on it!

The other $O(g^0)$ contribution to the distribution function comes from the term $A_t^{(0)} A_t^{(1)}$ in Eq. (32). Here $A_t^{(1)}$ includes the one loop radiative correction to the background field. One can show [15] from the operator equations of motion that $A_t^{(1)}$ satisfies the equation

$$\partial_-^2 A^{-,a} + D_i(\partial_- A_i^{(1),a}) + g f_{abc} \lim_{x \rightarrow y} \partial_{y-} G_{ii}^{bc}(x, y) = j_{cl}^a(x) \quad (35)$$

The final term on the left hand side is the induced current in the background field. We find that it has exactly the same structure as the classical current J_{cl}^+ and its only effect is to renormalize the external charge $\rho(x_t)$. It therefore also follows that $A_t^{(1)}$ has exactly the same structure as $A_t^{(0)}$. The only change is that

$$\alpha_S \longrightarrow \alpha_S(\mu^2). \quad (36)$$

We may therefore conclude that the only effect of the classical field $\langle A_t^{cl} \rangle$ is to replace the bare coupling constant in the ladder sum by a coupling constant which runs

as a function of μ^2 . This result is not entirely surprising [21] but it is nevertheless interesting to see how it arises in our formalism.

5. From virtual dream to hard reality

In the preceding sections, we investigated the properties of the NAWW field of a single, large nucleus. This is relevant for deeply inelastic scattering off nuclei at small x . To understand nuclear collisions, however, we need to understand how the NAWW fields evolve after the collision. In addition, we need to address the question of how the virtual partons go “on shell” and undergo hard scattering. The scattering is hard because most of the wee partons are in the “hard” kinematic region $\alpha_S \mu \ll k_t \ll \mu$. In the first part of this section, we will discuss ideas of Larry McLerran and collaborators at Minnesota on the *classical* problem of the evolution of the NAWW fields. The latter part will be a brief discussion of “onium–onium” scattering as formulated by A. H. Mueller.

5.1. The dynamical evolution of classical non–Abelian Weizsäcker–Williams fields in nuclear collisions

Recently, A. Kovner, L. McLerran and H. Weigert [22] managed to solve the *classical* problem of the evolution of these fields. They showed that at late times, the equations for the evolution of the fields are linear.

Below, we outline the important results in their paper and refer the inter-

ested reader to their paper for further details. The Yang-Mills equation for the two source problem is

$$D_\mu F^{\mu\nu} = J^\nu(x) \quad (37)$$

where

$$\begin{aligned} J^\pm &= \delta(x^\mp) \rho_{1,2}(x_\perp) \\ J^i &= 0. \end{aligned} \quad (38)$$

In the above, $\rho_1(x_t)$ and $\rho_2(x_t)$ are the color charge distributions of the valence quarks of the two nuclei.

Before the two nuclei collide (for times $t < 0$), the above equations of motion are satisfied by the classical field configuration where $A^\pm = 0$ and

$$A^i = \theta(x^-) \theta(-x^+) \alpha_1^i(x_\perp) + \theta(x^+) \theta(-x^-) \alpha_2^i(x_\perp) \quad (39)$$

The two dimensional vector potentials are pure gauges (as in the single nucleus problem!) and for $t < 0$ solve

$$\nabla \cdot \alpha_{1,2} = g^2 \rho_{1,2}(x_\perp). \quad (40)$$

The interesting aspect of this solution is that the classical field configuration does not evolve in time for $t < 0$! This is a consequence of the highly coherent character of the wee parton cloud in the nuclei.

The above solution for $t < 0$ is a fairly straightforward deduction from the single nucleus case. What is very interesting is that the above mentioned authors find a solution to the field equations for $t > 0$. It is given by

$$\begin{aligned} A^\pm &= \pm x^\pm \alpha(\tau, x_\perp) \\ A^i &= \alpha_3^i(\tau, x_\perp), \end{aligned} \quad (41)$$

where $\tau = \sqrt{t^2 - z^2} = \sqrt{2x^+ x^-}$. This solution only depends on the longitudinal

boost invariant variable τ and has no dependence on the space-time rapidity variable $y = \frac{1}{2} \ln \frac{x^+}{x^-}$. That the solution is independent of y suggests that the parton distributions will be boost invariant for all later times. This result therefore justifies Bjorken's ansatz [23] for the subsequent hydrodynamic evolution of the system.

The equations that result for $\alpha(\tau, x_\perp)$ and $\alpha_3(\tau, x_\perp)$ are highly non-linear. The detailed expressions are given in Ref.(22).

Asymptotically, for large τ ,

$$\begin{aligned} \alpha(\tau, x_\perp) &\rightarrow V(x_\perp) \epsilon(\tau, x_\perp) V^\dagger(x_\perp) \\ \alpha_3^i(\tau, x_\perp) &\rightarrow V(x_\perp) \left[\epsilon^i(\tau, x_\perp) - \frac{1}{ig} \partial^i \right] V^\dagger(x_\perp) \end{aligned} \quad (42)$$

where the value of the gauge transformation $V(x_\perp)$ is determined by the field equations. It results from solving the non-linear time evolution equations for the fields.

The equations of motion for the fields in the asymptotic region are linear for ϵ and ϵ^i . The solutions to these equations at asymptotically large τ are of the form

$$\begin{aligned} \alpha^a(\tau, x_\perp) &= \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{\sqrt{2\omega}} \left\{ a_1^a(\vec{k}_\perp) \frac{1}{\tau^{3/2}} e^{ik_\perp \cdot x_\perp - i\omega\tau} + C \right\} \\ \vec{\alpha}^{a,i}(\tau, x_\perp) &= \int \frac{d^2 k_\perp}{(2\pi)^2} \kappa^i \frac{1}{\sqrt{2\omega}} \left\{ a_2^a(k_\perp) \frac{1}{\tau^{1/2}} e^{ik_\perp \cdot x_\perp - i\omega\tau} + \right. \end{aligned}$$

In this equation, the frequency $\omega = |k_\perp|$, and the vector

$$\kappa^i = \epsilon^{ij} k^j / \omega \quad (44)$$

The notation $+C.C.$ means to add in the complex conjugate piece.

With the above form for the fields, the expressions for the number and energy densities of the partons is straightforward.

For late times, near $z = 0$, they look like [24]

$$dE = \frac{dz}{t} \int \frac{d^2 k_\perp}{(2\pi)^2} \omega \sum_{i,b} |a_i^b(k_\perp)|^2 \quad (45)$$

Recalling that $dy = dz/t$, we find that

$$\frac{dE}{dy d^2 k_\perp} = \frac{1}{(2\pi)^3} \omega \sum_{i,b} |a_i^b(k_\perp)|^2 \quad (46)$$

and the multiplicity distribution of gluons is

$$\frac{dN}{dy d^2 k_\perp} = \frac{1}{\omega} \frac{dE}{dy d^2 k_\perp} \quad (47)$$

The characteristic time scale for the dissipation of the non-linearities in the equations for the time dependent Weizsäcker-Williams fields can be estimated from dimensional arguments. This is of the order $\tau = 1/p_\perp \equiv 1/\alpha_S \mu$. Note that for large μ 's, i.e., large parton densities, this characteristic time gets smaller as one may intuitively expect.

5.2. Onium–onium scattering

An interesting re–formulation of the low x problem has been developed recently in a paper by A. H. Mueller [25]. He considers an “Onium” (heavy quark–anti-quark) state of mass M for which $\alpha_S(M) \ll 1$. In weak coupling, the n –gluon component of the onium wavefunction obeys an integral equation whose kernel in the leading logarithmic and large N_c limit is precisely the BFKL kernel we discussed in the introduction. The derivation relies on a picture in which the onium state produces a cascade of soft gluons strongly ordered in their longitudinal momentum; the i –th

emitted gluon has a longitudinal momentum much smaller than the $i - 1$ –th.

The procedure to compute the wave function can be recast in terms of hamiltonian perturbation theory in the following way: The Light Cone QCD Hamiltonian [10] can be split into two pieces: $P^- = P_0^- + P_{int}^-$, where P_0^- and P_{int}^- are the non–interacting and interacting pieces. Detailed expressions are given in Ref. (15).

In the Infinite Momentum frame, where the parton picture can be applied, the interaction Hamiltonian acting on an onium state produces a state with a quark–antiquark pair and an extra gluon. First order perturbation theory can be used to give an expression for that state according to

$$|\psi^{(1)}\rangle = \sum_{q_1, q_2} \frac{\langle q_1, q_2, p - q_1 - q_2 | P_{int}^- | k_1, p - k_1 \rangle}{E_{k_1}^{(0)}(2 - part) - E_{q_1, q_2}^{(0)}(3 - part)} |q_1, q_2, p -$$

where $E_{k_1}^{(0)}(n - part)$ is the eigenvalue of P_0^- on a state consisting of n free (quasireal) particles on mass shell; k_1 , $p - k_1$ are the quark and antiquark momenta and the sum is over all q_1, q_2 consistent with overall momentum conservation. Notice that the energy denominator in equation (48) will be equal to the (Light Cone) energy of the created gluon on mass shell, namely $k_{1t}^2/2k_1^+$, due to the strong ordering assumption. The state with n gluons and/or m quarks (anti-quarks) can be obtained from the vacuum by applying n and/or m times the corresponding creation operators appearing in the Light Cone definition of the fields in the usual

way,

$$A_i^\alpha(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2k^+}} \sum_{\lambda=1}^2 \epsilon_i(\lambda, k) \left[a^\alpha(\lambda, k) e^{ikx} + b^\alpha(\lambda, k) e^{-ikx} \right]$$

$$\Psi(x) = \int \frac{d^3k}{(2\pi)^3} \left[b(k) e^{ikx} + d^\dagger(k) e^{-ikx} \right].$$

The wave function in momentum space is obtained by projecting the state (48) onto the particular set of particle momenta. The gluon number, is obtained by squaring the wave function and adding its terms coherently. States with more than one gluon can be created by successively applying the same procedure to the already created states, and the gluon density to that order can be computed by squaring coherently the wave function so obtained.

In the large N_c limit, the n gluons can be represented as a collection of n -dipoles. Hence, in high energy onium-onium scattering, the cross section is proportional to the product of the number of dipoles in each onium state times the dipole-dipole scattering cross section [26]. This cross section is given by two gluon exchange [4] (the pomeron). More complicated exchanges are also possible.

It will be interesting to see if this approach can be extended to the problem of nuclear scattering—how does it relate to the approach in the first part of this section? That there are parallels between the two approaches is evident—even though the large N_c limit is a great simplification in the onium case.

6. Summary and outlook

In this work, we have discussed a many body approach to computing parton distribution functions for large nuclei for $x \ll$

$A^{-1/3}$. Knowledge of these structure functions is crucial in formulating the initial conditions for ultra-relativistic nuclear collisions. Furthermore, such a first principles calculation would eliminate the large uncertainties (especially at LHC) in jet cross sections and like rates [27]. One may also apply these methods to understand the systematics of nuclear shadowing in deeply inelastic scattering experiments. Much work needs to be done in these directions.

There are several open theoretical questions which need to be addressed. A pressing question is with regard to the applicability of weak coupling methods to large nuclei. Is $\mu^2 \sim A^{1/3}$ large enough? For the largest nuclei, we have at most $\mu^2 \sim 2 \text{ GeV}^2$. At first blush, this seems extremely unlikely. However, due to the possibility of a hierarchy of scales in the problem (in analogy to finite temperature QCD), such a conclusion may be unduly pessimistic. *If weak coupling methods are not applicable for computing the parton distributions of the largest nuclei, it is unlikely that they will ever apply during the subsequent stages of the evolution.* This would imply that a weakly interacting Quark Gluon Plasma would never be formed!

Another question we would like to address is whether the methods discussed for large nuclei can be extended to hadronic collisions. This depends on whether we can find a way to perform the quantum mechanical sum over the external sources in the partition function in Eq. (8).

The $x \rightarrow 0$ limit of QCD is a very interesting one because many features of the theory simplify in this limit. For instance, it has been shown recently that this limit,

for $N_c \rightarrow \infty$ is an exactly solvable theory [28]. It will be interesting to see whether these results can be recovered in the many body formalism discussed in this paper.

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